On the gaps of Neumann eigenvalues for Hill's equation with symmetric double well potential

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Abstract

A symmetric double well potential on [0,a] is defined as symmetric with respect to midpoint a/2 and quarter point a/4, also nonincreasing on [0,a/4]. In this study, some gaps between eigenvalues are minimized and maximized for Hill's equation with Neumann boundary conditions when the potential is symmetric double well.

2000 Mathematics Subject Classification. **34B30**. 34L15.
Keywords. Hill' s equation, symmetric double well potential, Neumann eigenvalues.

1 Introduction

We are concerned with Hill's equation

$$y''(t) + [\lambda - q(t)]\, y(t) = 0, \quad t \in [0, a] \tag{1.1}$$

where λ is a real parameter, $q(t)$ is a real-valued, continuous and periodic function with period a. This equation is explained with different boundary conditions. Equation (1.1) with boundary conditions $y(0) = y(a)$ and $y'(0) = y'(a)$ is defined as periodic problem. The eigenvalues of this problem are countable infinity and denoted by $\{\lambda_n\}$. Equation (1.1) with boundary conditions $y(0) = -y(a)$ and $y'(0) = -y'(a)$ is defined as semi-periodic problem or anti-periodic problem and denoted by $\{\mu_n\}$. Equation (1.1) with boundary conditions $y(0) = y(a) = 0$ is named by Dirichlet boundary value problem with eigenvalues $\{\Lambda_n\}$ and Equation (1.1) with boundary conditions $y'(0) = y'(a) = 0$ is named by Neumann boundary value problem with eigenvalues $\{\nu_n\}$.

We know the relationship between eigenvalues of Hills equation proven by [7] as following:

$$-\infty < \lambda_0 < \mu_0 \leq \mu_1 < \lambda_1 \leq \lambda_2 < \mu_2 \leq \mu_3 < ....$$

and for $n = 0, 1, 2, ...$

$$\mu_{2n} \leq \Lambda_{2n} \leq \mu_{2n+1}, \qquad \lambda_{2n+1} \leq \Lambda_{2n+1} \leq \lambda_{2n+2},$$

$$\mu_{2n} \leq \nu_{2n+1} \leq \mu_{2n+1}, \qquad \lambda_{2n+1} \leq \nu_{2n+2} \leq \lambda_{2n+2}. \tag{1.2}$$

Also, we should remark that for $\lambda \in (\lambda_{2n}, \mu_{2n}) \cup (\mu_{2n+1}, \lambda_{2n+1})$, all solutions of Hill's equation are bounded in $(-\infty, +\infty)$ and for $\lambda \in (-\infty, \lambda_0) \cup (\mu_{2n}, \mu_{2n+1}) \cup (\lambda_{2n+1}, \lambda_{2n+2})$, all nontrivial solutions of Hill's equation are unbounded in $(-\infty, +\infty)$. For this instability intervals of Hill's equation with symmetric potentials have been investigated by many authors. Some important studies of them are [1, 5, 6, 9, 10, 11].

Tbilisi Mathematical Journal Special Issue (8 - 2021), pp. 139–145.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 11 June 2020.
Accepted for publication: 06 November 2020.

A symmetric double well potential on $[0, a]$ is defined as symmetric with respect to midpoint $a/2$ and quarter point $a/4$, also nonincreasing on $[0, a/4]$. Double-well potential structures provide rich capabilities for single electron control via different physical characteristics of the potentials and the electron state. Especially in recent years, since quantum mechanic has gained importance, there are a lot of studies on eigenvalues of Hill's equation and Schrodinger operator with symmetric double well potential. [16] extends the analytical transfer matrix method to solve the energy splitting in an arbitrary symmetric double well potential. In [12], semiclassical periodic orbit theory is applied to the double-well eigenvalue problem to show how unified approach describes the quite different character of the level splitting (in the case of symmetric wells) and level shifts (in the asymmetric case) caused by tunneling. In [4], the effect of molecular symmetry on coherent tunneling in symmetric double-well potentials whose two molecular equilibrium configurations are interconverted by nuclear permutations is discussed. This is illustrated with vibrational tunneling in ammonia molecules, electronic tunneling in the dihydrogen cation, and laser-induced rotational tunneling of homonuclear diatomics. [15] investigates the tunneling dynamics of a three-dimensional cigar-shaped dipolar BoseEinstein condensate of ^{52}Cr atoms in an axially-symmetric double-well potential. [17] gives how the driving field and strong interatomic interaction affect the parity-time symmetry and stability of two interacting bosons in a non-Hermitian double-well system by means of a multiple-time-scale asymptotic analysis. [2] determines the effect of a uniform magnetic field on the electron state interference pattern manifesting in a focusing double-well potential structure by conducting Wigner quantum transport experiments. In [14], the spectral structure and many-body dynamics of two and three repulsively interacting bosons trapped in a one-dimensional double-well, for variable barrier height, inter-particle interaction strength, and initial conditions are examined. [3] and [10] are especially referred that $q(t)$ is a symmetric double well potential in Hill's equation. The eigenvalue gap for Schrodinger operators on an interval with Dirichlet and Neumann boundary conditions is considered in [3] and some results about the first instability interval are obtained in [10].

In this paper, we give the length between Neumann and the periodic eigenvalues, also the bounds of the length between Neumann and semi-periodic eigenvalues, when the potential is symmetric double well. We note that a symmetric double well potential on $[0, a]$ means a continuous function $q(t)$ on $[0, a]$ which is symmetric on $[0, a]$ as well as on $\left[0, \frac{a}{2}\right]$ and non-increasing on $\left[0, \frac{a}{4}\right]$ that is, $q(t) = q(a - t) = q(\frac{a}{2} - t)$ mathematically.

2 Main results

First of all, we remark that $q\prime(t)$ exists since a monotone function on an interval I is differentiable almost everywhere on I [8]. Our analysis is based on the following theorem of [3]:

The periodic and semi-periodic eigenvalues of Equation (1.1) satisfy, as $n \to \infty$

$$\begin{aligned}\begin{matrix}\lambda_{2n+1}\\ \lambda_{2n+2}\end{matrix} &= \frac{4\left(n+1\right)^2\pi^2}{a^2} \mp \frac{1}{\left(n+1\right)\pi}\left|\int_0^{a/4} q'\left(t\right)\sin\left[\frac{4(n+1)\pi}{a}t\right]dt\right| \\ &- \frac{a}{16\left(n+1\right)^2\pi^2}\left[aq^2\left(a\right) + 2a\int_0^{a/4} q(t)q'\left(t\right)dt - 8\int_0^{a/4} tq(t)q'\left(t\right)dt\right] \\ &+ o\left(n^{-2}\right)\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}\begin{matrix}\mu_{2n}\\ \mu_{2n+1}\end{matrix} &= \frac{(2n+1)^2\pi^2}{a^2} \\ &- \frac{a}{4(2n+1)^2\pi^2}\left[aq^2(a) + 2a\int_0^{a/4} q(t)q'(t)\,dt - 8\int_0^{a/4} tq(t)q'(t)\,dt\right] \\ &+ o\left(n^{-2}\right).\end{aligned} \tag{2.2}$$

The purpose of this study is to prove the following theorems:

Theorem 2.1. Let $q(t)$ be a symmetric double well potential on $[0,a]$. Then, the length between Neumann and the periodic eigenvalues as $n\to\infty$

$$\begin{aligned}\nu_{2n+1} - \lambda_{2n} &= \frac{(4n+1)\pi^2}{a^2} \\ &-\frac{1}{n\pi}\left|\int_0^{a/4} q'(t)\sin\left[\frac{4n\pi}{a}t\right]dt\right| \\ &-\left[\frac{a}{4(2n+1)^2\pi^2} - \frac{a}{16n^2\pi^2}\right] \\ &\times\left[aq^2(a) + 2a\int_0^{a/4} q(t)q'(t)\,dt - 8\int_0^{a/4} tq(t)q'(t)\,dt\right] \\ &+o\left(n^{-2}\right).\end{aligned}$$

Theorem 2.2. Let $q(t)$ be a symmetric double well potential on $[0,a]$. Then, the bounds for the length between Neumann and the semi-periodic eigenvalues as $n\to\infty$

$$\begin{aligned}\nu_{2n+2} - \mu_{2n+1} \geq \frac{(4n+3)\pi^2}{a^2} &- \frac{1}{(n+1)\pi}\left|\int_0^{a/4} q'(t)\sin\left[\frac{4(n+1)\pi}{a}t\right]dt\right| \\ &-\left[\frac{a}{16(n+1)^2\pi^2} - \frac{a}{4(2n+1)^2\pi^2}\right] \\ &\times\left[aq^2(a) + 2a\int_0^{a/4} q(t)q'(t)\,dt - 8\int_0^{a/4} tq(t)q'(t)\,dt\right] \\ &+ o(n^{-2})\end{aligned}$$

and

$$\begin{aligned}\nu_{2n+2}-\mu_{2n+1} \leq & \frac{(4n+3)\,\pi^2}{a^2}+\frac{1}{(n+1)\,\pi}\left|\int_0^{a/4} q'(t)\sin\left[\frac{4(n+1)\pi}{a}t\right]dt\right| \\ & -\left[\frac{a}{16\,(n+1)^2\,\pi^2}-\frac{a}{4\,(2n+1)^2\,\pi^2}\right] \\ & \times\left[aq^2(a)+2a\int_0^{a/4} q(t)q'(t)\,dt-8\int_0^{a/4} tq(t)q'(t)\,dt\right] \\ & +o(n^{-2}).\end{aligned}$$

Proof of Theorem 2.1: By Equation (1.2), we obtain the bounds for $\nu_{2n+1}-\lambda_{2n}$ that

$$\mu_{2n}-\lambda_{2n}\leq\nu_{2n+1}-\lambda_{2n}\leq\mu_{2n+1}-\lambda_{2n}. \tag{2.3}$$

By rewriting λ_{2n+2} from Equation (2.1) for $2n$, we get that

$$\begin{aligned}\lambda_{2n} = & \frac{4n^2\pi^2}{a^2}+\frac{1}{n\pi}\left|\int_0^{a/4} q'(t)\sin\left[\frac{4n\pi}{a}t\right]dt\right| \\ & -\frac{a}{16n^2\pi^2}\left[aq^2(a)+2a\int_0^{a/4} q(t)q'(t)\,dt-8\int_0^{a/4} tq(t)q'(t)\,dt\right]+o\left(n^{-2}\right).\end{aligned}$$

By using the last equation and Equation (2.2), we calculate that

$$\begin{aligned}\mu_{2n+1}-\lambda_{2n} = & \frac{(4n+1)\pi^2}{a^2}-\frac{1}{n\pi}\left|\int_0^{a/4} q'(t)\sin\left[\frac{4n\pi}{a}t\right]dt\right| \\ & -\left[\frac{a}{4(2n+1)^2\pi^2}-\frac{a}{16n^2\pi^2}\right] \\ & \times\left[aq^2(a)+2a\int_0^{a/4} q(t)q'(t)\,dt-8\int_0^{a/4} tq(t)q'(t)\,dt\right]+o\left(n^{-2}\right)\end{aligned}$$

and this also equals to $\mu_{2n+1}=\mu_{2n}$ from Equation (2.2). So from Equation (2.3), the lower and upper bounds are same for $\nu_{2n+1}-\lambda_{2n}$ and this proves the theorem.

Proof of Theorem 2.2: By Equation (1.2), we obtain the bounds for $\nu_{2n+2}-\mu_{2n+1}$ that

$$\lambda_{2n+1}-\mu_{2n+1}\leq\nu_{2n+2}-\mu_{2n+1}\leq\lambda_{2n+2}-\mu_{2n+1}. \tag{2.4}$$

By using Equation (2.2) and Equation (2.3), we calculate that

$$\lambda_{2n+1}-\mu_{2n+1}=\frac{(4n+3)\,\pi^2}{a^2}-\frac{1}{(n+1)\,\pi}\left|\int_0^{a/4}q'\,(t)\sin\left[\frac{4(n+1)\pi}{a}t\right]dt\right|$$
$$-\left[\frac{a}{16\,(n+1)^2\,\pi^2}-\frac{a}{4\,(2n+1)^2\,\pi^2}\right]$$
$$\times\left[aq^2\,(a)+2a\int_0^{a/4}q(t)q'\,(t)\,dt-8\int_0^{a/4}tq(t)q'\,(t)\,dt\right]$$
$$+\,o(n^{-2})$$

and

$$\lambda_{2n+2}-\mu_{2n+1}=\frac{(4n+3)\,\pi^2}{a^2}+\frac{1}{(n+1)\,\pi}\left|\int_0^{a/4}q'\,(t)\sin\left[\frac{4(n+1)\pi}{a}t\right]dt\right|$$
$$-\left[\frac{a}{16\,(n+1)^2\,\pi^2}-\frac{a}{4\,(2n+1)^2\,\pi^2}\right]$$
$$\times\left[aq^2\,(a)+2a\int_0^{a/4}q(t)q'\,(t)\,dt-8\int_0^{a/4}tq(t)q'\,(t)\,dt\right]$$
$$+\,o(n^{-2}).$$

The last equations and Equation (2.4) prove the theorem.

3 An example

$q(x) = cos2x$ is important function and used with different types in Schrodinger equations as potential. This function is one of the examples of symmetric double well potential. Besides, Equation (1.1) with $q(x) = \varepsilon cos2x$ is known as Mathieu Equation. Here ε is a real parameter. Mathieu equation occurs in a broad spectrum of physical problems, for example, when the wave equation is separated using elliptic coordinates. Also, it describes the motion in periodic potentials, like the trajectory of an electron in an array of atoms, the mechanics of the quantum pendulum, and the motion of charged particles inside a quadrupole mass-filter [13]. So, if we rewrite our conclusions for $q(x) = \varepsilon cos2x$ on $[0, 2\pi]$, we reach the following results, as $n \to \infty$:

$$\nu_{2n+1}-\lambda_{2n}=\frac{4n+1}{4}-\left[\frac{\varepsilon^2}{2\,(2n+1)^2}-\frac{\varepsilon^2}{8n^2}\right]+o\left(n^{-2}\right),$$

$$\nu_{2n+2}-\mu_{2n+1}=\frac{4n+3}{4}-\left[\frac{\varepsilon^2}{8\,(n+1)^2}-\frac{\varepsilon^2}{2\,(2n+1)^2}\right]+o\left(n^{-2}\right)$$

The last equality holds because integral term $\left|\int_0^{a/4}q'\,(t)\sin\left[\frac{4(n+1)\pi}{a}t\right]dt\right|$ is zero for $q(x) = \varepsilon cos2x$ and $a = 2\pi$, so the upper bound equals the lower bound in Theorem 2.2.

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